Cubic Fuzzy Precision: Tau-scaled Fractals from Plain-vanilla Pentagrids

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Abstract

The discoveries of Penrose tiles, quasicrystals, and Penrose/pentagrid duality have breathed new life into 5-fold symmetries. But in the quest for aperiodicity, cases the layman would call exotic are deemed "regular" and the plain-vanilla cases are deemed "singular", even "exceptionally singular", and have been left largely unexplored (paradoxically, they might be the most computationally challenging [1]). "Fuzzy precision" uses a mechanical approach to explore the case of perfectly periodic grids superposed on an "exceptionally singular" origin to find surprisingly rich metastructure in the patterns that emerge from the ranked near-misses of incommensurable integer/irrational line-crossings. "Cubic fuzzy precision" extends the explorations to 3D. Since these are in fact limit cases of the periodic/aperiodic landscape, perhaps they can aid in broader generalizations.

Genesis of the Exploration

The term "cubic fuzzy precision" might at first seem odd. It reflects the genesis of a multi-stage exploration over time. "Fuzzy precision" evokes the method of analyzing the imprecision of the intersections of 2D grids in the plane. "Cubic" evokes the application of the method to a 3D mesh.

Moiré macrostructures. The exploration was triggered by seeing the shifting moiré effects produced by rotating 2 identical triangular grids *(Fig.1)*.

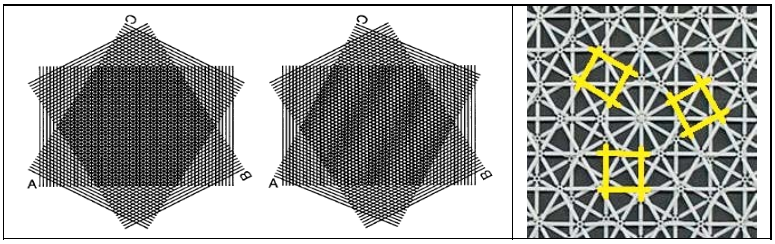

Figure 1: *3 identical line grids 120°apart = no moiré. Small angle change= prominent moiré.*
2 triangular grids are functionally equivalent to 3 square grids or 6 line-grids.
This hexagrid is one type of "multigrid" (6 sets of parallel lines).
Also explored are tetragrids (4 line-grids) and pentagrids (5 line-grids).

Incommensurable irrationals from pure periodicity. A grid of equilateral triangles is structured by 2 lengths: triangle sides of s=1, and triangle heights of h=√3/2. If 2 triangular grids are overlaid on a shared center and rotated 30° relative to each other, then on the resulting multigrid, the sides and heights series are superposed on a line radiating from the central origin. This is the *only* point the s and h series will share. The 2 series can be extended to infinity without the lengths ever again coinciding since √3/2 is irrational, but will converge ever more as we move from the origin, e.g. 30 units of h=√3/2 (≈25.980) closely approximates 26 units of s=1.

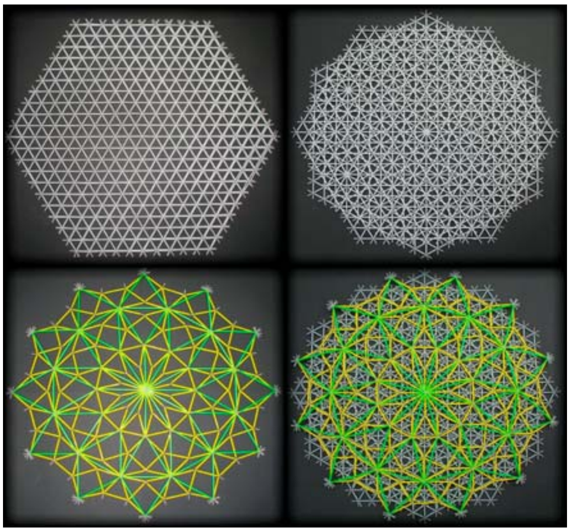

Figure 2: *Hexagrid (2 triangular grids): aligned; rotated. Fractal tiling: unmounted; mounted on hexagrid.*

Similar precisions form a √3-scaled fractal mesh. The rotation creates 12 identical radials, on which the 12 first-convergence points are closer to each other than to the origin. On a compass arc lie points of neighboring precision on 12 "intermediate" radials interleaved among the initial 12 radials. We identify 72 similar points (including 36 on intermediate radials) which interlink in a mesh formed of equilateral triangle, square, and rhombus -- each of equal sides and angles in multiples of 30° *(Fig.2)*. The mesh is predetermined and non-periodic. Proceeding to the next level of lesser precision, we find a smaller (scale factor 2+√3 ≈3.732) fractal version (*Fig.3*). See Vienne [3] for table of convergents; see Hausman [4] for video descriptions in English and French.

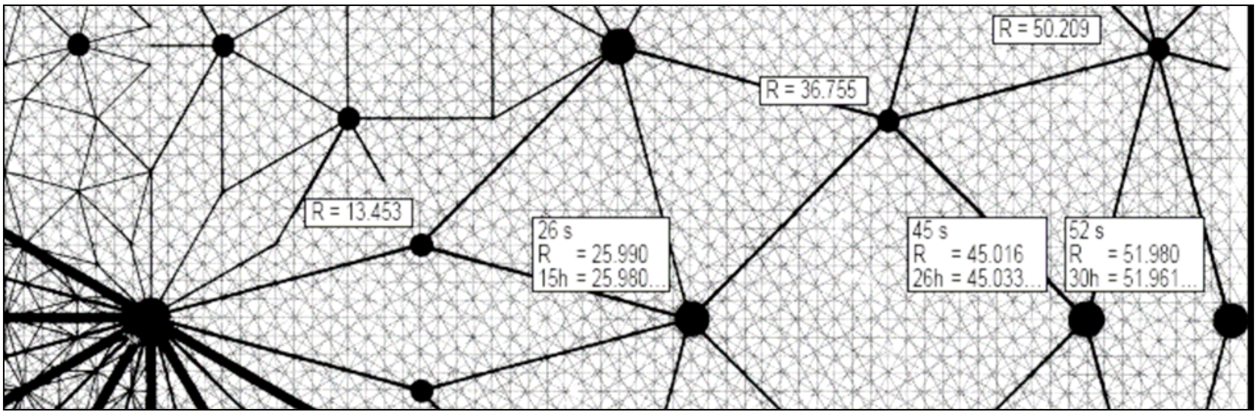

Figure 3: 3 *convergence points on 1 of 12 primary radials, interlinked with convergence points on 1 of 12 intermediate radials. (s= triangle side, h= triangle height, R= rosette node)*

Tetragrid and resulting fractal mesh. Next applied to 2 square grids, this method gave similar results with values *s*=1 and *h*=√2 for the series of square-sides and square-diagonals, a more rapid Greek-ladder convergence (see Osler [5]) to the third decimal at 17*s* for 12*h*, and a mesh having as components a square and a rhombus with angles in multiples of 45°. (*Fig.4*)

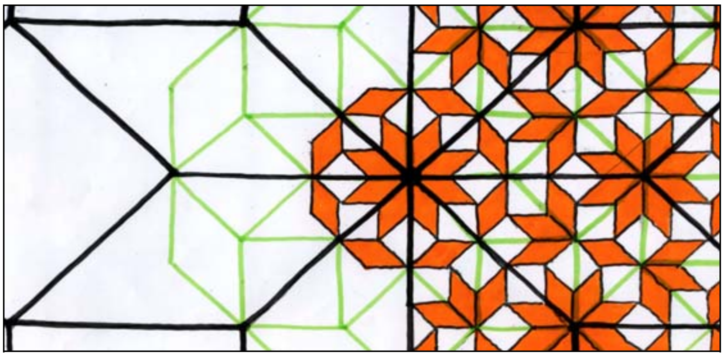

Figure 4: *Tetragrid and resulting mesh.*

Pentagrid: By generalization, it appears that this method could apply to all meshes formed by multigrids of equidistant parallels in a plane. See Schoen [2] for a unified theory of rosettes. Hence the idea to transpose it to 3D, by superposing cubic frames. Grouped about an origin in dodecahedral symmetry, 5 cubic grids have 5 networks of parallel planes, separated by the edge-length of a cube, intergrouped in each of the 6 projection planes of this symmetry. Beginning from the point of origin, each projection plane taken separately contains 10 radials in the direction of the cube edges perpendicular to the parallel planes, joining them at different angles. These intersections determine length units that are related by the golden ratio tau (an irrational) to the cube edge of U=1. For the radials determined by the cube edges, there are 3 lengths: 1, 1/phi•2, phi•2. For the intermediary radials, there are 2 lengths: 2/√phi+2 and phi•2/√phi +2, converging in an increasing Fibonacci series towards tau, which is ubiquitous in this structure. In each plane, the mesh formed of these intersections is composed of a pentagon, a starred pentagon, and 2 rhombi, all with angles in multiples of 36 degrees -- with the criteria of non-periodicity and fractality that correspond, on each level, to the similar-precision intersections of the cubic grids. (*Fig.5*)

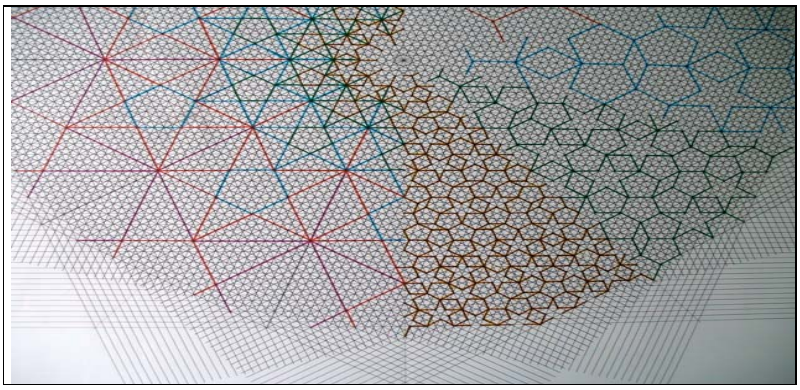

Figure 5: *Pentagrid and resulting mesh.*

Pentagrids extended to 3D: In space, the 10 radials of each plane situated at the intersection of these planes brings to 30 the number of identical radials from the center through the vertices of an icosidodecahedron. They define 60 pizza-slices of identical mesh passing through the edges of the icosidodecahedron and 2 types of "carrots" or cone-shaped volumes passing through the icosidodeca faces: 12 carrots of pentagonal cross-section and 20 carrots of triangular cross-section. These volumes are filled by an alternation of interpenetrating dodecahedra and icosahedra -- interlinked concentrically and laterally by starred pentagons -- which expand radially to infinity. Thus, the mesh is reproduced in parallel in the 6 identical projection planes dividing the space that corresponds, at each level of the progression, to the similar-precision intersections of the cubic grids (*Fig.6*). A section of this network -- between two concentric icosidodecahedral shells -- has been designed as a "Pentigloo kiosk" for the Zometool barnraising at Bridges2011.

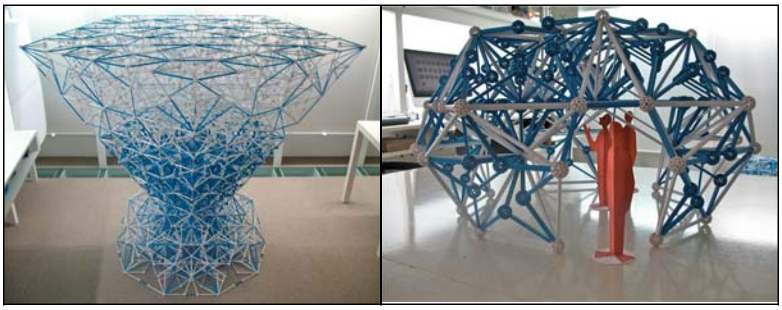

Figure 6: *One hemisphere of the icosidodecahedral core plus one pentagonal "carrot". Pentigloo kiosk for 2011 Zometool barnraising.*

Concluding remarks: The mesh intersections place themselves at the centroid of the near-miss intersections of the base grids. The greater the distance from the origin, the greater the precision of the convergents crossings, and the greater therefore their resemblance to the origin. The origin is omnipresent and so is infinity.

For further exploration:

Simple replacement in Theon's ladder (Osler [5]) to converge on roots and sequences like Fibonacci might mimic natural processes more closely than Newton's or other computational methods. Converging even more slowly than Fibonacci, could the Padovan sequence be a number "more irrational" than tau for computations in phyllotaxis or other natural processes?

The $\sqrt{3}$-based hexagrid seems to show a "sweet spot" at 26 units and the seventh level of recursion. Could this reflect the hurdle in stellar nucleosynthesis at the atomic weight of iron? Do patterns of sweet spots mimic the periodic table's pattern of magic numbers?

Now that it is known that Penrose rhomb tilings are dual to pentagrids and isomorphic to decagonal coverings with matching overlaps (Schoen [2]), it might pay to explore singular and exceptionally singular (unshifted) pentagrids to see if they exhibit properties thought to be limited to aperiodic "regular" grids (shifted radially or by interval).

References

[1] N. G. de Bruijn, *Nederl. Akad. Wetensch. Indag. Math*. 43, 39–52, 53–66 (1981). Algebraic theory of Penrose's nonperiodic tilings of the plane, I, II

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[5] Osler, Thomas J. , Wright, Marcus and Orchard, Michael(2005) "Theon's ladder for any root", *International Journal of Mathematical Education in Science and Technology*, 36: 4, 389-398.